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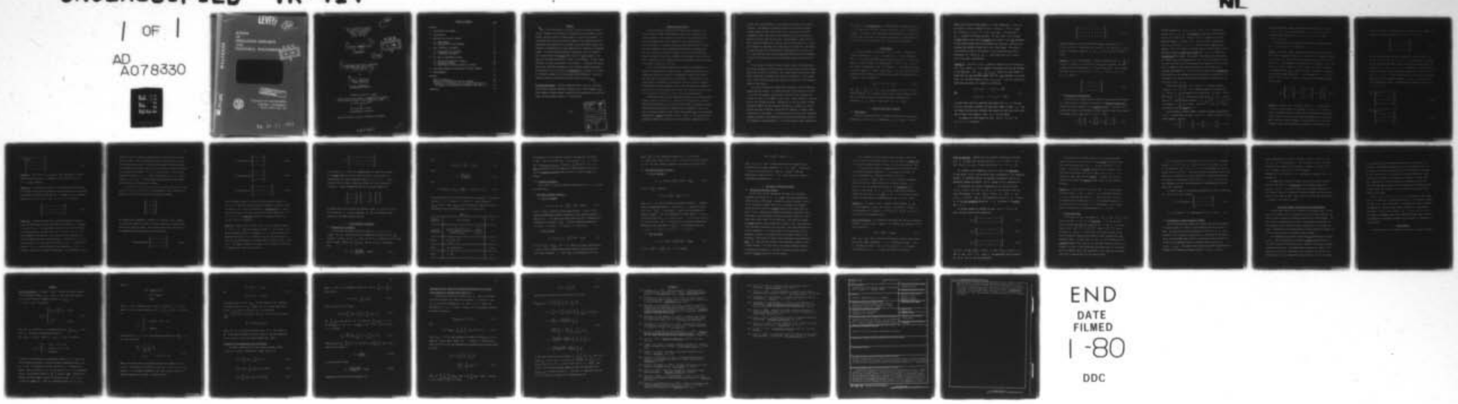

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INCOMPLETE BLOCK DESIGNS FOR COMPARING TREATMENTS WITH A CONTROL (I): GENERAL THEORY

by

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ABSTRACT

This is the first paper of a series in which we outline a theory of optimal incomplete block designs for comparing several treatments with a control, and give plans of optimal designs to use for this purpose. In the present paper we develop the basic theory of the incomplete block designs which are appropriate for the many-to-one comparisons problem; the optimality considerations for such designs will be addressed in subsequent papers. For this problem we propose a new general class of incomplete block designs which are balanced with respect to test treatments. We shall use the abbreviation BTIB to refer to such designs. We study their structure, and give some methods of construction. A procedure for making exact joint confidence statements for our multiple comparisons problem is described. Using a new concept of admissibility of designs, it is shown how "inferior" designs can be eliminated from consideration. Some open problems concerning construction of BTIB designs are posed.

Key words and phrases: Multiple comparisons with a control, incomplete block designs, balanced treatment incomplete block (BTIB) designs, admissible designs, optimal designs, applications of equicorrelated multivariate normal and multivariate Student's t distributions.

1. INTRODUCTION AND SUMMARY

In many industrial, agricultural and biological experiments it is often desired to compare simultaneously several test treatments with a control treatment. The earliest correct work on this problem was carried out by Dunnett [8], [9]. Dunnett [8] also posed (but did not solve) the problem of optimally allocating experimental units to control and test treatments so as to maximize the probability associated with the joint confidence statement concerning the many-to-one comparisons between the mean of the control treatment and the means of the test treatments. Bechhofer and his coworkers solved this optimal allocation problem in a series of papers [1], [2], [3].

In all of the aforementioned papers it was tacitly assumed that a completely randomized (CR) design was used. However, many practical situations may require the blocking of treatments in order to cut down on bias and improve the precision of the experiment. If the block size is large enough to accomodate one replication of all of the test treatments and additional control treatments as well, then the design and analysis of replications of the experiment can be carried out using the optimal allocations described in [1] and [2] with only the usual modifications.

We shall study the multiple comparisons problem in the situation which commonly occurs in practice, i.e., when all of the blocks have a common size but the block size is less than the total number of treatments. Robson [21] pointed out that Dunnett's procedure can be extended to the case in which a balanced incomplete block (BIB) design between <u>all</u> of the treatments (including the <u>control</u> treatment) is used. Cox [7, p. 238] noted that BIB designs are perhaps not appropriate for the multiple comparisons with the

control (MCC) problem because of the special role played by the control treatment. He suggested a design which employs the control treatment an equal number of times (once, twice, etc.) in each block, the test treatments forming a BIB design in the remaining plots of the blocks; no analytical details were given for this proposed design. Pešek [20] has given analytical details for a special case of Cox's design (i.e., the control treatment is employed once in each block); he shows that this design is more efficient than a BIB design for comparisons with a control but it is less efficient for pairwise comparisons between the test treatments. It should be noted that even Cox's more general design is quite restrictive.

In our series of papers we shall propose a theory of optimal incomplete block designs for the MCC problem, and give plans of optimal designs which would be useful in practice. The present paper deals with the basic theory underlying our designs; the problem of choosing an optimal design from a set of contending admissible designs is addressed in the subsequent papers of this series.

In Section 2 we give the linear model considered, and the underlying assumptions and notation. We define the class of BTIB designs in Section 3, and find the necessary and sufficient conditions that a design must satisfy in order to be a member of this class. Several methods of construction of such designs are given. Expressions for the best linear unbiased estimators (BLUE's) of the treatment differences of interest for the MCC problem are given in Section 4 along with the analysis of variance table associated with our design; it is shown how one can obtain joint confidence interval estimates of the differences of interest using existing tables of the integral of the multivariate t or the multivariate normal distribution.

The concept of the admissibility of a BTIB design is introduced in Section 5; it is shown there how this criterion can be used to rule out inferior designs. In Section 6 some remarks are offered concerning the content of future papers, and several open problems are mentioned. Mathematical details of the results in Section 3 and 4 are given in the Appendix.

2. PRELIMINARIES

Let the treatments be indexed by $0,1,\ldots,p$ with 0 denoting the control treatment and $1,2,\ldots,p$ denoting the $p \geq 2$ test treatments. Let $k < p+1$ denote the common size of each block, and let b denote the number of blocks available for experimentation. Thus $N = kb$ is the total number of experimental units. If treatment i is assigned to the hth plot of the jth block $(0 \leq i \leq p,\ 1 \leq h \leq k,\ 1 \leq j \leq b)$, let Y_{ijh} denote the corresponding random variable; we assume the usual additive linear model (no treatment $\times$ block interaction)

$$Y_{ijh} = \mu + \alpha_i + \beta_j + e_{ijh} \tag{2.1}$$

with $\sum_{i=0}^{p} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$; the e_{ijh} are assumed to be i.i.d. $N(0,\sigma^2)$ random variables. It is desired to make an exact joint confidence statement (employing one-sided or two-sided intervals) concerning the p differences $\alpha_0 - \alpha_i$ based on their BLUE's $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$.

3. CHOICE OF THE CLASS OF DESIGNS

3.1 BTIB designs

Since it is desired to make a confidence statement which applies simultaneously to all of the p differences $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$, we shall

regard our problem as being symmetric in these differences. To this end we consider a class of designs for which $\mathrm{Var}\{\hat{\alpha}_0-\hat{\alpha}_i\} = \eta^2\sigma^2/N$ $(1\leq i\leq p)$ and $\mathrm{Corr}\{\hat{\alpha}_0-\hat{\alpha}_{i_1}, \hat{\alpha}_0-\hat{\alpha}_{i_2}\} = \rho$ $(i_1\neq i_2;\ 1\leq i_1,i_2\leq p)$; the parameters η and ρ depend on the design employed. We shall refer to such designs as BTIB designs since they are balanced with respect to the test treatments. The following theorem states the necessary and sufficient conditions that a design must satisfy in order that it be a BTIB design. The proof of this theorem is given in the Appendix; in the process we also derive expressions for $\mathrm{Var}\{\hat{\alpha}_0-\hat{\alpha}_i\}$ and ρ. These quantities play a crucial role in our later considerations.

Theorem 3.1: For given (p,k,b) consider a design with the incidence matrix $\{r_{ij}\}$ where r_{ij} is the number of replications of the ith treatment in the jth block. Let $\lambda_{i_1i_2} = \sum_{j=1}^{b} r_{i_1j}r_{i_2j}$ denote the total number of times that the i_1th treatment appears with the i_2th treatment in the same block over the whole design $(i_1\neq i_2;\ 0\leq i_1,i_2\leq p)$. Then the necessary and sufficient conditions for a design to be BTIB are:

$$\lambda_{01} = \lambda_{02} = \dots = \lambda_{0p} = \lambda_0 \quad \text{(say)}$$

and (3.1)

$$\lambda_{12} = \lambda_{13} = \dots = \lambda_{p-1,p} = \lambda_1 \quad \text{(say)}.$$

In other words, each test treatment must appear with (i.e., in the same block as) the control treatment the same total number of times (λ_0) over the design, and each test treatment must appear with each other test treatment the same total number of times (λ_1) over the design.

An example of a BTIB design for which $(p,k,b) = (4,3,10)$ and $\lambda_0 = 4$, $\lambda_1 = 2$ is given by

$$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 3 & 3 \\ 2 & 2 & 3 & 4 & 3 & 4 & 3 & 4 & 4 & 4 \end{matrix}\right\}. \tag{3.2}$$

Additional examples of some selected BTIB designs are given in (3.3)-(3.10), (5.2)-(5.4). Expressions for $\mathrm{Var}\{\hat{\alpha}_0-\hat{\alpha}_i\}$ and $\mathrm{Corr}\{\hat{\alpha}_0-\hat{\alpha}_{i_1}, \hat{\alpha}_0-\hat{\alpha}_{i_2}\}$ $(i_1\neq i_2)$ are given in terms of λ_0 and λ_1 by (4.2) and (4.4), respectively.

Remark 3.1: We note that Theorem 3.1 places no restriction on $r_i = \sum_{j=1}^{b} r_{ij}$ $(1\leq i\leq p)$, the number of replications of the ith test treatment, and hence a design can be BTIB without the r_i $(1\leq i\leq p)$ being equal. Such a design for which $(p,k,b) = (4,3,7)$ and $\lambda_0 = 2$, $\lambda_1 = 2$ with $r_1 = r_2 = r_3 = 4$, $r_4 = 5$ is given by

$$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 4 \\ 3 & 4 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}. \tag{3.3}$$

3.2 Construction of BTIB designs

At this point we indicate several methods of constructing BTIB designs. As a starting point we introduce the concept of a minimal BTIB generator design (more simply referred to as a generator design). For given (p,k) a generator design is a BTIB design no proper subset of whose blocks forms a BTIB design; here at least one of $\lambda_0,\lambda_1 > 0$. Thus

$$D_0 = \left\{\begin{matrix} 0 & 0 \\ 1 & 2 \end{matrix}\right\}, \quad D_1 = \left\{\begin{matrix} 1 \\ 2 \end{matrix}\right\}, \quad D = \left\{\begin{matrix} 0 & 0 & 1 \\ 1 & 2 & 2 \end{matrix}\right\} \tag{3.4}$$

are BTIB designs with $(\lambda_0, \lambda_1) = (1,0), (0,1), (1,1)$, respectively; however, only D_0 and D_1 are generator designs (as are the designs given by (3.2) and (3.3)). Design D of (3.4) suggests the role of generator designs. For given (p,k) there are several generator designs; e.g., for each $p \geq 2$, $k = 2$ there are exactly two generator designs. (See (3.5), below.) By taking unions of replications of these generator designs, at least one of which has $\lambda_0 > 0$, we obtain an implementable BTIB design. The problem of determining how many generator designs exist for arbitrary given (p,k) is an open problem (it can be shown that there are a finite number) which appears to be very formidable; we comment on this problem in Section 6, and consider it in Bechhofer and Tamhane [5]. In the sequel we consider only implementable BTIB designs; we also assume that no block contains only one of the $p+1$ treatments since clearly such a block contributes no information to the estimation of the $\alpha_0 - \alpha_i$.

Suppose that for given (p,k) there are n generator designs D_i $(1 \leq i \leq n)$. Let $\lambda_0^{(i)}, \lambda_1^{(i)}$ be the design parameters associated with D_i, and let b_i be the number of blocks required by D_i $(1 \leq i \leq n)$. Then a BTIB design $D = \bigcup_{i=1}^{n} f_i D_i$ obtained by forming unions of $f_i \geq 0$ replications of D_i has the design parameters $\lambda_0 = \sum_{i=1}^{n} f_i \lambda_0^{(i)}$, $\lambda_1 = \sum_{i=1}^{n} f_i \lambda_1^{(i)}$ and requires $b = \sum_{i=1}^{n} f_i b_i$ blocks. The set of D_i with $f_i > 0$ will be referred to as the support of D.

The following is another example of generator designs (generalizing (3.4)). As noted above, for each $p \geq 2$, $k = 2$ there are exactly two generator designs. They are

$$D_0 = \left\{\begin{matrix} 0 & 0 & \cdots & 0 \\ 1 & 2 & & p \end{matrix}\right\}, \quad D_1 = \left\{\begin{matrix} 1 & 1 & \cdots & p-1 \\ 2 & 3 & & p \end{matrix}\right\}. \tag{3.5}$$

From these generator designs, implementable BTIB designs of the type $D = f_0D_0 \cup f_1D_1$ can be constructed for $f_0 \geq 1$, $f_1 \geq 0$: the corresponding design parameters for D are $\lambda_0 = f_0$, $\lambda_1 = f_1$ while $b = f_0p + f_1p(p-1)/2$.

We now consider several methods of constructing generator designs for $k \geq 3$; this list is not exhaustive.

Method I: The above example suggests the following method of constructing a class of generator designs: For given (p,k), a generator design D_m will have $m+1$ plots in each block assigned to the control treatment: the p test treatments are assigned to the remaining $k-m-1$ plots of the b_m blocks $(0 \leq m \leq k-2)$ in such a way as to form a BIB design. The generator design D_{k-1} contains no control treatments: it consists of a RB (BIB) design between the p test treatments if $p = k$ $(p>k)$. Thus for $(p,k) = (3,3)$ we have the following three generator designs in this class.

$$D_0 = \left\{\begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{matrix}\right\}, \quad D_1 = \left\{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{matrix}\right\}, \quad D_2 = \left\{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right\}. \tag{3.6}$$

Method II: Starting with a BIB design between $t > p$ treatments in b blocks, one can relabel the treatments $p+1, p+2, \ldots, t$ by zeros to obtain a new design which is a BTIB design with possibly an additional block or blocks, each one of the latter containing only one test treatment or only the control treatment. After deleting all of these one-treatment blocks, and identifying the support of the resulting BTIB

design, one obtains the desired generator design(s). As an example we consider the following BIB design for $(t,k,b) = (7,3,7)$:

$$\begin{Bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \end{Bmatrix} . \tag{3.7}$$

By replacing the sevens by zeros one obtains a generator design for $(p,k,b) = (6,3,7)$ with $\lambda_0 = 1$, $\lambda_1 = 1$. This process can be continued by then replacing the sixes by zeros to obtain a generator design for $(p,k,b) = (5,3,7)$ with $\lambda_0 = 2$, $\lambda_1 = 1$; continuing, one can then replace the fives by zeros to obtain a generator design for $(p,k,b) = (4,3,7)$ with $\lambda_0 = 3$, $\lambda_1 = 1$. Finally, replacing the fours by zeros one obtains the union of the two generator designs D_0 and D_1 of (3.6) with a block containing all zeros. The BTIB designs obtained in this way for $k = 3$, $b = 7$ are

<u>$p = 6$, $b = 7$</u> $(\lambda_0 = 1, \lambda_1 = 1)$

$$\begin{Bmatrix} 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 2 & 5 & 3 & 4 \\ 3 & 6 & 5 & 4 & 6 & 5 & 6 \end{Bmatrix} \tag{3.8a}$$

<u>$p = 5$, $b = 7$</u> $(\lambda_0 = 2, \lambda_1 = 1)$

$$\begin{Bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 3 & 4 & 0 & 2 & 3 \\ 3 & 5 & 4 & 5 & 2 & 4 & 5 \end{Bmatrix} \tag{3.8b}$$

$p = 4,\ b = 7$ $(\lambda_0 = 3,\ \lambda_1 = 1)$

$$\left\{\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 2 \\ 3 & 3 & 4 & 1 & 2 & 4 & 4 \end{array}\right\}. \tag{3.8c}$$

Remark 3.2: From (3.8a) it is clear that every BIB design (or Youden square design) involving t treatments yields a BTIB design with $p = t-1$ test treatments.

Remark 3.3: It is well-known (see, e.g., John [15]) that unequal replicate designs having the same properties as BIB designs can be constructed. Such designs are therefore BTIB designs. An example of such a generator design for $(p,k,b) = (5,3,8)$ with $\lambda_0 = 0$, $\lambda_1 = 2$ is given by

$$\left\{\begin{array}{cccccccc} 1 & 1 & 1 & 2 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 & 5 & 5 & 5 & 5 \\ 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 \end{array}\right\}. \tag{3.9}$$

Method III: Consider a group divisible partially balanced incomplete block (GD-PBIB) design with two associate classes between t treatments in blocks of size k. The association scheme of such a GD-PBIB design can be represented in the form of an $m \times n$ array (with $mn = t$). Any two treatments in the same row of the array are first associates and those in different rows are second associates. Suppose that $m \geq k$; one can then take $p = m$ and relabel the entries in $n_1 > 0$ columns of the array by $1,2,\ldots,p$ and the entries in the remaining $n_2 = n-n_1 > 0$ columns by zeros, thus obtaining a BTIB design. As with Method II, such a

design may not be a generator design and may contain some blocks which must be deleted. After deleting such blocks a BTIB design is obtained. By identifying the support of this resulting design the desired generator design(s) are obtained; some (or all) of these can usually be obtained by the previous two methods. If $n \geq k$ one can take $p = n$ and then relabel the entries in $m_1 > 0$ rows of the array by $1,2,\ldots,p$ and the entries in $m_2 = m-m_1 > 0$ rows by zeros, thus obtaining a BTIB design, possibly with blocks which must be deleted.

To illustrate the use of this method consider the GD-PBIB design #R20 (for k=3, t=12, m=4, n=3, b=36) in the monograph by Bose, Clatworthy and Shrikhande [6] which has the following association scheme:

$$\left\{\begin{matrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{matrix}\right\}.$$

By relabeling the treatments 5 through 8 by 1 through 4, and 9 through 12 by zeros one obtains the union of a BTIB design with a design containing one block with only zeros. After deleting that block the support of the remaining BTIB design consists of the following:

$$2 \text{ replications of } \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 4 \end{matrix}\right\}, \qquad (3.10a)$$

$$2 \text{ replications of } \left\{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix}\right\}, \tag{3.10b}$$

$$2 \text{ replications of } \left\{\begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \end{matrix}\right\}, \tag{3.10c}$$

$$1 \text{ replication of } \left\{\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 & 3 \\ 2 & 4 & 4 & 3 & 3 & 4 & 4 \end{matrix}\right\}. \tag{3.10d}$$

Thus the designs given by (3.10a)-(3.10d) are generator designs for $p = 4$, $k = 3$; the first three designs are obtainable by Method I while the fourth is not. Using the definition of admissibility of a design as given in Section 5, we shall see that this fourth design with $b = 7$, $\lambda_0 = 2$, $\lambda_1 = 2$ is admissible relative to the design obtained for $(p,k,b) = (4,3,7)$ by Method II.

Method IV: Suppose that for given (p,k) we have a generator design D_1 with $\lambda_0 > 0$. Then a new generator design D_2 for the same (p,k) can be obtained by taking a "complement" of D_1 in the following way: Separate the blocks of D_1 in different sets so that each block in a given set has zero assigned in an equal number of plots (0 times, 1 time, etc.). For example, consider the design (3.8c) the blocks of which can be separated into three sets as follows:

$$D_1 = \left\{\begin{array}{c:ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 3 & 0 & 0 & 0 \\ 4 & 3 & 3 & 4 & 1 & 2 & 4 \end{array}\right\}.$$

For each set of D_1 write its "complementary" set (with zero assigned in the same number of plots) so that the union of that set with its complementary set forms a generator design; if $r_{ij} = 0$ or 1 $(1 \leq i \leq p)$ then that union is simply a generator design that can be constructed by Method I. These complementary sets in the present example are

$$\left\{\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 4 \end{array}\right\}, \left\{\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 4 & 4 \end{array}\right\}, \left\{\begin{array}{c} 0 \\ 0 \\ 3 \end{array}\right\};$$

by taking their union we obtain the generator design D_2 given by (3.10d). The b-values for D_1 and its complement D_2 are not in general equal, although in the present example they are.

4. JOINT CONFIDENCE STATEMENTS

4.1 Expressions for estimates

We first give the expressions for the BLUE $\hat{\alpha}_0 - \hat{\alpha}_i$ of $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$. Let T_i denote the sum of all observations obtained with the ith treatment $(0 \leq i \leq p)$, and let B_j denote the sum of all observations in the jth block $(1 \leq j \leq b)$. Define $B_i^* = \sum_{j=1}^{b} r_{ij} B_j$ and let $Q_i = kT_i - B_i^*$ $(0 \leq i \leq p)$. Then

$$\hat{\alpha}_0 - \hat{\alpha}_i = \frac{\lambda_1 Q_0 - \lambda_0 Q_i}{\lambda_0(\lambda_0 + p\lambda_1)} \quad (1 \leq i \leq p). \tag{4.1}$$

Also,

$$\mathrm{Var}\{\hat{\alpha}_0-\hat{\alpha}_i\} = \frac{\eta^2}{N}\sigma^2 \quad (1\leq i\leq p) \tag{4.2}$$

where

$$\eta^2 = \frac{k^2 b(\lambda_0+\lambda_1)}{\lambda_0(\lambda_0+p\lambda_1)}, \tag{4.3}$$

and

$$\rho = \mathrm{Corr}\{\hat{\alpha}_0-\hat{\alpha}_{i_1},\ \hat{\alpha}_0-\hat{\alpha}_{i_2}\} = \frac{\lambda_1}{\lambda_0+\lambda_1}, \quad (i_1\neq i_2;\ 1\leq i_1, i_2\leq p). \tag{4.4}$$

The expressions for (4.1)-(4.4) are derived in the Appendix. An unbiased estimate s_ν^2 of σ^2 based on $\nu = N-p-b$ d.f. can be computed as $SS_{error}/(N-p-b)$ where SS_{error} can be obtained by subtraction (as in BIB designs) from the following analysis of variance table.

Table 4.1

Source of Variation	Sum of Squares	d.f.
Treatments (adjusted)	$\dfrac{Q_0^2\{(p-1)(\lambda_0^2+\lambda_1^2)+(p^2+3)\lambda_0\lambda_1\}}{k(p+1)^2\lambda_0^2(\lambda_0+p\lambda_1)} + \dfrac{\sum_{i=1}^{p} Q_i^2}{k(\lambda_0+p\lambda_1)}$	p
Blocks	$\dfrac{1}{k}\sum_{j=1}^{b} B_j^2 - \dfrac{G^2}{N}$	$b-1$
Error	(by subtraction)	$N-p-b$
Total	$H - \dfrac{G^2}{N}$	$N-1$

The expressions in the table are derived in the Appendix; the symbols G and H are also defined there. We note that if $\lambda_0 = \lambda_1 = \lambda$ (say), then $SS_{\text{treatments(adjusted)}}$ reduces to $\sum_{i=0}^{p} Q_i^2/k\lambda(p+1)$, i.e., the same expression as for a BIB design; this latter expression thus holds for any completely balanced design (such as (3.3) which is not a BIB design).

4.2 Confidence statements

Joint $100(1-\alpha)$ percent confidence intervals for the $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$ are given below.

I. One-sided confidence intervals

a) For σ^2 unknown:

$$\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - t_{\nu,p,\rho}^{(\alpha)} s_\nu \eta/\sqrt{N} \quad (1 \leq i \leq p) \tag{4.6}$$

In (4.6) $t_{\nu,p,\rho}^{(\alpha)}$ denotes the upper equicoordinate α point of the p-variate equicorrelated central t-distribution with common correlation ρ, and with d.f. ν (as defined by Dunnett and Sobel [10]). Krishnaiah and Armitage [17] have tabled $t_{\nu,p,\rho}^{(\alpha)}$ to three significant figures for $p = 1(1)10$; $\nu = 5(1)35$; $\alpha = 0.05, 0.01$; $\rho = 0.0(0.1)0.9$.

b) For σ^2 known:

$$\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - z_{p,\rho}^{(\alpha)} \sigma\eta/\sqrt{N} \quad (1 \leq i \leq p) \tag{4.7}$$

In (4.7) $z_{p,\rho}^{(\alpha)}$ $(= t_{\nu,p,\rho}^{(\alpha)}$ for $\nu = \infty)$ denotes the upper equicoordinate α point of the p-variate equicorrelated standard normal distribution with common correlation ρ. Gupta, Nagel and Panchapakesan [13] have

tabled $z_{p,\rho}^{(\alpha)}$ to five significant figures for $p = 1(1)10(2)50$; $\alpha = 0.250, 0.100, 0.050, 0.025, 0.010$; $\rho = 0.1(0.1)0.9, 0.125(0.125)0.875, 1/3, 2/3$. Two other relevant references are Gupta [12] and Milton [19].

II: Two-sided confidence intervals

a) For σ^2 unknown:

$$\alpha_0 - \alpha_i \in [\hat{\alpha}_0 - \hat{\alpha}_i \pm t'^{(\alpha)}_{\nu,p,\rho} \; s_\nu \eta/\sqrt{N}] \quad (1 \leq i \leq p) \tag{4.8}$$

In (4.8) $t'^{(\alpha)}_{\nu,p,\rho}$ satisfies

$$P\{|t_i| \leq t'^{(\alpha)}_{\nu,p,\rho} \quad (1 \leq i \leq p)\} = 1 - \alpha$$

where $(t_1, \ldots, t_p)$ has a p-variate equicorrelated central t distribution with common correlation ρ, and with d.f. ν (Dunnett and Sobel [10]), i.e., $t'^{(\alpha)}_{\nu,p,\rho}$ is the upper α point of the distribution of $\max\{|t_i| \; (1 \leq i \leq p)\}$. Hahn and Hendrickson [14] have tabled t' to four significant figures for $p = 1(1)6(2)12,15,20$; $\nu = 3(1)12,15(5)30,40,60$; $\alpha = 0.10,0.05,0.01$, and $\rho = 0.0(0.2)0.4,0.5$. See also, Krishnaiah and Armitage [18] who have tabled $(t'^{(\alpha)}_{\nu,p,\rho})^2$ to two decimal places for $p = 1(1)10$; $\nu = 5(1)35$; $\alpha = 0.05,0.01$; $\rho = 0.0(0.1)0.9$.

b) For σ^2 known:

$$\alpha_0 - \alpha_i \in [\hat{\alpha}_0 - \hat{\alpha}_i \pm z'^{(\alpha)}_{p,\rho} \sigma\eta/\sqrt{N}] \quad (1 \leq i \leq p) \tag{4.9}$$

In (4.9) $z'^{(\alpha)}_{p,\rho}$ ($= t'^{(\alpha)}_{\nu,p,\rho}$ for $\nu = \infty$) satisfies

$$P\{|Z_i| \leq z'^{(\alpha)}_{p,\rho} \quad (1 \leq i \leq p)\} = 1 - \alpha$$

where $(Z_1,\ldots,Z_p)$ has a p-variate equicorrelated standard normal distribution with common correlation ρ, i.e., $z'^{(\alpha)}_{p,\rho}$ is the upper α point of the distribution of $\max\{|Z_i| \ (1 \leq i \leq p)\}$. Hahn and Hendrickson's tables [14] with $\nu = 60$ can be used here to obtain a conservative approximation to $\nu = \infty$.

5. THE CLASS OF ADMISSIBLE DESIGNS

5.1 Optimal and admissible designs

We have shown how the experimenter can make joint confidence statements concerning the $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$ when the experiment was conducted using a BTIB design. However we have not proposed a rationale for choosing one design from a set of contending BTIB designs. Toward that end we define an optimal design as that design which for given (p,k,b) maximizes the appropriate confidence coefficient (for one-sided or two-sided common confidence intervals) subject to a specified limit on the "width" of the intervals. (One might also define an optimal design as that design which for given (p,k) and specified joint confidence coefficient (for one-sided or two-sided confidence intervals) minimizes b, the total number of blocks required for the experiment, subject to a specified limit on the common "width" of the intervals.) The solution to both of these problems can be obtained for known σ^2 (since then the "width" of the intervals is fixed, and not a random variable). We shall focus on one-sided confidence intervals in Bechhofer and Tamhane [4], [5], and in those papers shall give the plans for optimal designs for the MCC problem.

In the remainder of this section we shall describe a simple rule which can be used to determine whether or not one design is <u>inadmissible</u> w.r.t. another design for given (p,k,b), in the sense described below.

To illustrate the concept of inadmissibility we consider the case of one-sided confidence intervals; σ^2 is assumed to be known. We limit consideration to confidence intervals of the form $\{\alpha_0-\alpha_i \geqq \hat{\alpha}_0-\hat{\alpha}_i \geqq d \quad (1\leqq i\leqq p)\}$ where $d > 0$ is a specified "yardstick" associated with the common width of the confidence intervals. For given (p,k,b) we say that BTIB design D_2 is <u>inadmissible</u> with respect to BTIB design D_1 if D_1 yields a larger joint confidence coefficient than does D_2 for <u>every</u> d and σ. The following theorem gives a characterization of inadmissibility which is easy to verify.

<u>Theorem 5.1</u>: For given (p,k,b) consider two BTIB designs D_1 and D_2 with parameters (η_1^2,ρ_1) and (η_2^2,ρ_2), respectively. Design D_2 is inadmissible w.r.t. design D_1 if and only if $\eta_1^2 \leqq \eta_2^2$ and $\rho_1 \geqq \rho_2$ with at least one inequality strict.

<u>Proof of sufficiency</u>: The probability associated with the joint confidence statement $\{\alpha_0-\alpha_i \geqq \hat{\alpha}_0-\hat{\alpha}_i-d \quad (1\leqq i\leqq p)\}$ for a design with parameters (η^2,ρ) can be written as

$$P\{Z_i < d\sqrt{N}/\sigma\eta \quad (1\leqq i\leqq p)\} \tag{5.1}$$

where $(Z_1,\ldots,Z_p)$ has a p-variate equicorrelated standard normal distribution with common correlation ρ. As η decreases for fixed d and ρ (ρ increases for fixed d and η) the probability (5.1) increases. (The monotonicity w.r.t. ρ follows from Slepian's inequality.)

Proof of necessity: Suppose that the confidence coefficient associated with D_1 is larger than that associated with D_2 for every d and σ. Then $\eta_1^2 \leq \eta_2^2$ $(\rho_1 \geq \rho_2)$ follows from letting $d \uparrow \infty$ $(d \downarrow 0)$. QED

If a design is not inadmissible then it is said to be admissible. For any given (p,k,b) one would then limit consideration to all admissible designs. As indicated earlier, the problem of enumerating this set of admissible designs for general (p,k,b) appears to be a formidable one.

To illustrate the concept of admissibility we consider the case $(p,k,b) = (3,3,3)$ for which D_0 and D_1 of (3.6) are two of the possible BTIB designs; D_0 has $(\eta_0^2,\rho_0) = (81/10,\ 1/3)$ while D_1 has $(\eta_1^2,\rho_1) = (27/2,0)$. Thus D_1 is inadmissible relative to D_0. Actually D_0 is the only admissible design for $b = 3$, and hence it is optimal for $b = 3$.

As another example we consider the case $(p,k,b) = (3,2,9)$ for which the only possible BTIB designs are

$$D_0 = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{matrix}\right\} \tag{5.2a}$$

$$D_1 = \left\{\begin{matrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{matrix}\right\} \tag{5.2b}$$

$$D_2 = \left\{\begin{matrix} 1 & 1 & 2 & 1 & 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 3 & 2 & 3 & 3 & 1 & 2 & 3 \end{matrix}\right\}. \tag{5.2c}$$

In (5.2), D_0 has $(\eta_0^2,\rho_0) = (12,0)$, D_1 has $(\eta_1^2,\rho_1) = (108/10,\ 1/3)$, and D_2 has $(108/7,\ 2/3)$. Thus D_0 is inadmissible being dominated by D_1; D_1 and D_2 are both admissible.

The admissibility criterion described in the paragraphs above applies equally well to the case of joint two-sided confidence intervals; then the monotonicity w.r.t. ρ follows from Šidák's [23] results. (Of course, the optimal designs might be different in the one-sided and two-sided cases for the same (p,k,b) and d.) The same general ideas carry over for unknown σ^2 except that then one would have to specify the expected common "width" of the confidence intervals.

Remark 5.1: We note from (4.3) and (4.4) that η^2 is a decreasing function of both λ_0 and λ_1 while ρ is a decreasing function of λ_0 but an increasing function of λ_1. Thus if for given (p,k,b) two designs have the same values of λ_0 but different values of λ_1, then the design with the smaller λ_1 is inadmissible. This phenomenon is illustrated by the inadmissibility of D_1 in (3.6).

5.2 Equivalent designs

If for given (p,k,b) two designs D_1 and D_2 have $(\eta_1^2,\rho_1)=(\eta_2^2,\rho_2)$, then we say that D_1 and D_2 are equivalent. It is easy to verify that a necessary and sufficient condition for D_1 and D_2 to be equivalent is that $(\lambda_0^{(1)},\lambda_1^{(1)}) = (\lambda_0^{(2)},\lambda_1^{(2)})$ where $(\lambda_0^{(i)},\lambda_1^{(i)})$ refers to design D_i $(i=1,2)$. A trivial case of equivalent designs arises when one design is obtained from another by simply permuting the test treatment labels. In such a case D_1 and D_2 will be referred to as isomorphic designs; this case is of no practical interest, and we do not consider it further. A more interesting case arises when the two designs are not isomorphic and yet are equivalent; in our papers the term equivalent will be reserved only for the latter designs.

As an illustration of such designs for $(p,k,b) = (4,3,7)$ consider designs D_2 of (3.3) and D_3 (say) of (3.10d). The designs D_2 and D_3 are equivalent since $(\lambda_0,\lambda_1) = (2,2)$ for both designs; but the de-designs are not isomorphic. Equivalent designs can sometimes provide flexibility to the experimenter without changing the confidence coefficient. For example, the experimenter might prefer D_3 to D_2 if (say) the control treatments are more readily available for experimentation than any of the test treatments.

As another example of two equivalent designs which are not isomorphic, for $(p,k,b) = (6,3,35)$ consider

$$D_1 = 5 \text{ replicates of } \left\{\begin{matrix} 0 & 0 & 0 & 1 & 2 & 3 & 1 \\ 1 & 2 & 4 & 2 & 3 & 4 & 5 \\ 3 & 6 & 5 & 4 & 5 & 6 & 6 \end{matrix}\right\} \tag{5.3}$$

$$D_2 = \{\text{all distinct } \binom{7}{3} \text{ combinations of } (0,1,\ldots,6)\}. \tag{5.4}$$

5.3 Relationship to other optimality criteria

Kiefer (along with many others) has studied extensively the problem of optimal design in a series of papers starting with the one [16] in 1958. The three main criteria considered by Kiefer are D-, A- and E-optimality which correspond, respectively, to minimizing the determinant, trace, and the maximum eigen value of the variance-covariance matrix of the BLUE of the parameter vector of interest which in our case is $(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_p)'$. The eigen values (neglecting constant proportionality factors) of the variance-covariance matrix of $(\hat{\alpha}_0-\hat{\alpha}_1,\ldots,\hat{\alpha}_0-\hat{\alpha}_p)'$

can be obtained from (4.2)-(4.4); they are $(\lambda_0+p\lambda_1)^{-1}$ of multiplicity $p-1$, and λ_0^{-1} of multiplicity 1. Thus for given (p,k,b) the three criteria can be stated as a) D-optimality: minimize $\{\lambda_0(\lambda_0+p\lambda_1)^{p-1}\}^{-1}$, i.e., maximize $\{\lambda_0(\lambda_0+p\lambda_1)^{p-1}\}$; b) A-optimality: minimize$\{\lambda_0^{-1} + (p-1)(\lambda_0+p\lambda_1)^{-1}\}$; c) E-optimality: minimize λ_0^{-1}, i.e., maximize λ_0.

Although these criteria are simpler than ours it should be kept in mind that they refer to an ellipsoidal joint confidence region for $(\alpha_0-\alpha_1,\ldots,\alpha_0-\alpha_p)'$. In the case of MCC such a confidence region is of less interest than the "rectangular" confidence regions that we have proposed, and are commonly used. Hence we do not consider the Kiefer criteria further in this series of papers.

6. CONCLUDING REMARKS AND DIRECTIONS OF FUTURE RESEARCH

In the present paper we have introduced a new general class of incomplete block designs which are appropriate for use in the MCC problem. We refer to these as balanced treatment incomplete block (BTIB) designs. The basic results concerning the structure of such designs are derived, and the properties of the relevant estimates obtained with such designs are given. Admissibility and inadmissibility of these designs are defined, and these criteria are used to eliminate inferior designs.

The combinatorial problem of constructing all BTIB designs for given (p,k,b), and the procedure for choosing an optimal design from such a set are not solved in the present paper. However, some methods of design construction are given. The aforementioned problems are related in the sense that to solve the optimization part completely one must have constructed most if not all generator designs for given (p,k); the problem

of determining how many generator designs exist for arbitrary (p,k), and then of enumerating them appears to be a very formidable one. Alternatively, the problem of construction of BTIB designs for arbitrary (p,k,b) can be set up in the manner of Foody and Hedayat [11, Lemma 4.1] which is suitable for a solution on a computer. However, there is no guarantee that all BTIB designs can be generated in this way. Also, the magnitude of difficulty of our problem is substantially greater than theirs, and therefore a solution via this route looks rather remote at this stage.

For $p \geq 2$, $k = 2$ and $p = 3$, $k = 3$ the situation is very simple since it is necessary to consider essentially only two generator designs for each (p,k) case; these cases are considered in detail in Bechhofer and Tamhane [4], and the optimal design is given for a large range of the useful (p,k,b)- and d-values. For $p = 4,5,6$ and $k = 3,4$ it is possible to enumerate most of the generator designs; these cases will be considered in Bechhofer and Tamhane [5] along with the associated optimal designs.

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APPENDIX

Proof of Theorem 3.1. For given (p,k,b) consider an arbitrary design with the incidence matrix $\{r_{ij}\}$. Then it is well known (see equation (3.1) of [16]) that the information matrix $\underset{\sim}{M} = \{m_{i_1 i_2}\}$ of $\underset{\sim}{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_p)'$ is given by

$$m_{i_1 i_2} = \begin{cases} r_{i_1} - \frac{1}{k} \sum_{j=1}^{b} r_{i_1 j}^2 & (i_1 = i_2) \\ \\ -\frac{\lambda_{i_1 i_2}}{k} & (i_1 \neq i_2). \end{cases} \tag{A.1}$$

Note that $\underset{\sim}{M}$: $(p+1)\times(p+1)$ is a singular matrix and $\sum_{i_2=0}^{p} m_{i_1 i_2} = 0$ for each i_1. We require the information matrix of $\underset{\sim}{U}\underset{\sim}{\alpha} = (\alpha_0 - \alpha_1, \ldots, \alpha_0 - \alpha_p)'$ where $\underset{\sim}{U} = \{u_{i_1 i_2}\}$: $p \times (p+1)$ is given by

$$u_{i_1 i_2} = \begin{cases} 1 & (i_2 = 1,\ i_1 = 1, \ldots, p) \\ -1 & (i_2 = i_1 + 1,\ i_1 = 1, \ldots, p) \\ 0 & \text{otherwise.} \end{cases}$$

In order to avoid computing the generalized inverse of $\underset{\sim}{M}$ we shall follow the method given below to obtain the desired information matrix. Let $\underset{\sim}{Q}$: $p \times (p+1)$ be any matrix the rows of which form p orthogonal contrasts. Then we can write $\underset{\sim}{U}$ as $\underset{\sim}{U} = \underset{\sim}{P}\underset{\sim}{Q}$ where $\underset{\sim}{P}$: $p \times p$ is a nonsingular matrix. The information matrix of $\underset{\sim}{Q}\underset{\sim}{\alpha}$ is given by $\underset{\sim}{Q}\underset{\sim}{M}\underset{\sim}{Q}'$. Therefore the variance-covariance matrix (except for the common factor σ^2) of $\underset{\sim}{U}\hat{\underset{\sim}{\alpha}}$ is given by $\underset{\sim}{P}(\underset{\sim}{Q}\underset{\sim}{M}\underset{\sim}{Q}')^{-1}\underset{\sim}{P}'$. Hence the information matrix $\underset{\sim}{M}^*$ of $\underset{\sim}{U}\underset{\sim}{\alpha}$ is

given by

$$\begin{aligned} M^* &= [P(QMQ')^{-1}P']^{-1} \\ &= (P')^{-1}(QMQ')P^{-1} \qquad \text{(A.2)} \\ &= V'MV \end{aligned}$$

where $V = Q'P^{-1}$ satisfies $UV = I$ with I being the $p \times p$ identity matrix. It can be easily verified that $V = \{v_{i_1 i_2}\}: (p+1) \times p$ is given by

$$v_{i_1 i_2} = \begin{cases} 0 & (i_1 = i_2 + 1, \quad i_2 = 1, \ldots, p), \\ 1 & \text{otherwise.} \end{cases}$$

Substituting for V in (A.2), the information matrix $M^* = \{m^*_{i_1 i_2}\}$ of $U\alpha$ can be written as

$$\begin{aligned} m^*_{i_1 i_2} &= \sum_{\substack{g=0 \\ g \neq i_1}}^{p} \sum_{\substack{h=0 \\ h \neq i_2}}^{p} m_{gh} \\ &= m_{i_1 i_2} \qquad (i_1, i_2 = 1, \ldots, p) \qquad \text{(A.3)} \end{aligned}$$

where (A.3) follows from the fact that the rows and columns of M sum to zero. For a design to be BTIB, the matrix M^* must be completely symmetric, i.e., all diagonal elements of M^* must be equal, and all off-diagonal elements must be equal. Therefore we have

$$m_{11} = m_{22} = \ldots = m_{pp}$$

and (A.4)

$$m_{12} = m_{13} = \ldots = m_{p-1,p}.$$

Using expression (A.1) for $m_{i_1 i_2}$ in (A.4) implies (3.1). Denoting the common value of the λ_{0i} $(1 \leq i \leq p)$ by λ_0, and the common value of the $\lambda_{i_1 i_2}$ $(i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p)$ by λ_1 we see that $m_{i_1 i_2} = -\lambda_1/k$ $(i_1 \neq i_2;\ 1 \leq i_1, i_2 \leq p)$ and $m_{ii} = \{\lambda_0 + (p-1)\lambda_1\}/k$ $(1 \leq i \leq p)$. Thus

$$\underset{\sim}{M}^* = \{(\lambda_0 + p\lambda_1)\underset{\sim}{I} - \lambda_1 \underset{\sim}{J}\}/k$$

where $\underset{\sim}{J}: p \times p$ is a matrix consisting only of 1's. The inverse of $\underset{\sim}{M}^*$ is the desired variance-covariance matrix of $\underset{\sim}{U}\hat{\underset{\sim}{\alpha}}$; the expressions for (4.2) and (4.4) are then easily obtained from $(\underset{\sim}{M}^*)^{-1}$.

Derivation of the expressions for the estimates

The normal equations for the least squares estimates (BLUE) $\hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j$ of μ, α_i, β_j, respectively, $(0 \leq i \leq p,\ 1 \leq j \leq b)$ are:

$$N\hat{\mu} + \sum_{i=0}^{p} r_i \hat{\alpha}_i + k \sum_{j=1}^{b} \hat{\beta}_j = G, \tag{A.6}$$

$$r_i \hat{\mu} + r_i \hat{\alpha}_i + \sum_{j=1}^{b} r_{ij} \hat{\beta}_j = T_i \quad (0 \leq i \leq p) \tag{A.7}$$

$$k\hat{\mu} + \sum_{i=0}^{p} r_{ij} \hat{\alpha}_i + k\hat{\beta}_j = B_j \quad (1 \leq j \leq b) \tag{A.8}$$

where T_i and B_j are defined in Section 4.1, and $G = \sum_{i=0}^{p} T_i = \sum_{j=1}^{b} B_j$. Substituting

$$\hat{\mu} + \hat{\beta}_j = (B_j - \sum_{i=0}^{p} r_{ij}\hat{\alpha}_i)/k \tag{A.9}$$

from (A.8) into (A.7) we obtain

$$r_i\hat{\alpha}_i + \frac{1}{k}\sum_{j=1}^{b} r_{ij}B_j - \frac{1}{k}\sum_{h=0}^{p}\hat{\alpha}_h\sum_{j=1}^{b} r_{hj}r_{ij} = T_i. \tag{A.10}$$

Now $\sum_{j=1}^{b} r_{ij}B_j = B_i^*$ and for $h \neq i$ we have that $\sum_{j=1}^{b} r_{hj}r_{ij} = \lambda_0\ (\lambda_1)$ if only one of h and $i = 0$ (if both h and $i \neq 0$). From (A.10) for $i = 0$ we obtain

$$r_0\hat{\alpha}_0 + \frac{B_0^*}{k} - \frac{1}{k}\{\hat{\alpha}_0\sum_{j=1}^{b} r_{0j}^2 + \lambda_0\sum_{h=1}^{p}\hat{\alpha}_h\} = T_0. \tag{A.11}$$

Using the fact that $\sum_{h=1}^{p}\hat{\alpha}_h = -\hat{\alpha}_0$ and that $r_0 - \frac{1}{k}\sum_{j=1}^{b} r_{0j}^2 = m_{00} = p\lambda_0/k$, from (A.11) we obtain

$$\hat{\alpha}_0 = \frac{Q_0}{(p+1)\lambda_0}\ . \tag{A.12a}$$

In the same way we obtain

$$\hat{\alpha}_i = \frac{Q_i + (\lambda_0 - \lambda_1)\hat{\alpha}_0}{(\lambda_0 + p\lambda_1)}\quad (1 \leq i \leq p). \tag{A.12b}$$

Combining (A.12a) and (A.12b) we obtain (4.1).

Derivation of the formula for the adjusted treatment sum of squares in the analysis of variance table (Table 4.1)

Following the Scheffé [22] notation, let S_Ω denote the minimum error sum of squares (SS) under the assumptions Ω of Section 2, and let S_ω denote the minimum error SS under $\omega = H \cap \Omega$ where the hypothesis is $H: \alpha_i = 0$ $(0 \leq i \leq p)$. Then the SS of treatments adjusted for blocks is given by

$$SS_{\text{treat(adj.)}} = S_\omega - S_\Omega. \tag{A.13}$$

Now

$$S_\Omega = SS_{\text{error}} = \sum_{i=0}^{p} \sum_{j=1}^{b} \sum_{h=1}^{k} (y_{ijh} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j)^2 I_{ijh} \tag{A.14}$$

where $I_{ijh} = 1$ if the $i\underline{th}$ treatment is assigned to the $h\underline{th}$ plot of the $j\underline{th}$ block $(0 \leq i \leq p,\ 1 \leq j \leq b,\ 1 \leq h \leq k)$ and $= 0$ otherwise. Substituting $\hat{\beta}_j$ from (A.9) into (A.14), and expanding and simplifying the resulting square we obtain

$$S_\Omega = H - \frac{1}{k} \sum_{j=1}^{b} B_j^2 + \sum_{i=0}^{p} r_i \hat{\alpha}_i^2$$

$$- \frac{1}{k} \sum_{j=1}^{b} \left(\sum_{i=0}^{p} r_{ij} \hat{\alpha}_i \right)^2 - Q \tag{A.15}$$

where $H = \sum_{i=0}^{p} \sum_{j=1}^{b} \sum_{h=1}^{k} y_{ijh}^2 I_{ijh}$ and $Q = \frac{2}{k} \sum_{i=0}^{p} \hat{\alpha}_i Q_i$. Under ω we have $\hat{\mu} = G/N$ and $\hat{\beta}_j = B_j/k - \hat{\mu}$; hence

$$S\omega = H - \frac{1}{k}\sum_{j=1}^{b} B_{j}^{2}. \qquad (A.16)$$

Subtracting (A.15) from (A.16) we have from (A.13) that

$$\begin{aligned}
SS_{treat(adj)} &= Q + \frac{1}{k}\sum_{j=1}^{b}\Big(\sum_{i=0}^{p} r_{ij}\hat{\alpha}_i\Big)^2 - \sum_{i=0}^{p} r_i\hat{\alpha}_i^2 \\
&= Q - \sum_{i=0}^{p}\Big(r_i - \frac{1}{k}\sum_{j=1}^{b} r_{ij}^2\Big)\hat{\alpha}_i^2 + \frac{2}{k}\sum_{i_1=0}^{p}\sum_{i_2=i_1+1}^{p} \hat{\alpha}_{i_1}\hat{\alpha}_{i_2}\lambda_{i_1 i_2} \\
&= Q - \frac{p\lambda_0}{k}\hat{\alpha}_0^2 - \frac{\{\lambda_0+(p-1)\lambda_1\}}{k}\sum_{i=1}^{p}\hat{\alpha}_i^2 \\
&\quad + \frac{2\lambda_0\hat{\alpha}_0}{k}\sum_{i=1}^{p}\hat{\alpha}_i + \frac{2\lambda_1}{k}\sum_{i_1=1}^{p}\sum_{i_2=i_1+1}^{p}\hat{\alpha}_{i_1}\hat{\alpha}_{i_2} \qquad (A.17)\\
&= Q - \frac{(p+2)\lambda_0\hat{\alpha}_0^2}{k} - \frac{(\lambda_0+p\lambda_1)}{k}\sum_{i=1}^{p}\hat{\alpha}_i^2 + \frac{\lambda_1}{k}\Big(\sum_{i=1}^{p}\hat{\alpha}_i\Big)^2 \\
&= Q - \frac{[(p+2)\lambda_0-\lambda_1]\hat{\alpha}_0^2}{k} - \frac{(\lambda_0+p\lambda_1)}{k}\sum_{i=1}^{p}\hat{\alpha}_i^2.
\end{aligned}$$

In the above we have used the relations $r_i - \frac{1}{k}\sum_{j=1}^{b} r_{ij}^2 = m_{ii}$ (from (A.1)) where $m_{00} = p\lambda_0/k$, $m_{ii} = \{\lambda_0 + (p-1)\lambda_1\}/k$ $(1 \leq i \leq p)$, and $\sum_{i=1}^{p}\hat{\alpha}_i = -\hat{\alpha}_0$. Substituting in (A.17) for $\hat{\alpha}_0$ from (A.12a) and for $\hat{\alpha}_i$ $(1 \leq i \leq p)$ from (A.12b), and after some tedious algebra we obtain the expression for $SS_{treat(adj)}$ as given in Table 4.1. The other SS expressions and the df in the table are obtained in a straightforward way.

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This is the first paper of a series in which we outline a theory of optimal incomplete block designs for comparing several treatments with a control, and give plans of optimal designs to use for this purpose. In the present paper we develop the basic theory of the incomplete block designs which are appropriate for the many-to-one comparisons problem; the optimality considerations for such designs will be addressed in subsequent papers. For this problem we propose a new general class of incomplete block designs which are balanced with respect to

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test treatments. We shall use the abbreviation BTIB to refer to such designs. We study their structure, and give some methods of construction. A procedure for making exact joint confidence statements for our multiple comparisons problem is described. Using a new concept of admissibility of designs, it is shown how "inferior" designs can be eliminated from consideration. Some open problems concerning construction of BTIB designs are posed.